The Picard group in equivariant homotopy theory via stable module categories

$Pic(Sp^G)$, describe in terms of iterated pullback diagrams involving $Pic(StMod(\mathbb{Z}W_G(K)))$

I. Equivariant homotopy theory

$\mathcal{S}^G$: ∞-cat. of G-spaces up to equivariant homotopy (there are fixed points $(-)^H: \mathcal{S}^G \to \mathcal{S}$)

Thm (Elmendorf):

$$\mathcal{S}^G = Fun(Orb(G)^{op}, \mathcal{S})$$

Sp^G: stabilizing $\mathcal{S}^G$ by inverting representation spheres $S^V = V \cup \{\infty\}$. stable, sym. monoidal ∞-cat. Here, finite G-CW complexes become dualizable (i.e., have equivariant S-duality)

$(-)^H: Sp^G \to Sp$, $(map_{Sp^G}(\Sigma^\infty_+ G/H, -))$

$\mathbb{S}^G$ has Ω^∞ $\operatorname{colim}_V Map^G(S^V, S^V)$

$\Phi^H: Sp^G \to Sp$

(i) $\Phi^H \Sigma^\infty_+ X = \Sigma^\infty_+ X^H$

(ii) commutes w/ (filtered) colimits

(iii) symmetric monoidal

II. Picard groups

For a symm. monoidal ∞-cat,

$$Pic(\mathcal{C}) = \text{equiv. classes of } \otimes\text{-invertible objects} = \pi_0 \mathcal{P}ic(\mathcal{C})$$

Ex: $Pic(Sp) = \mathbb{Z}$, $n \leftrightarrow \mathbb{S}^n$

$$\pi_i \mathcal{P}ic(Sp) = \begin{cases} \{\pm 1\} & i = 1 \\ \pi_{i-1}(\mathbb{S}) & i \geq 2 \end{cases}$$

Question: $Pic(Sp^G)$

$\Sigma^\infty S^V$ are invertible objects

$$RO(G) \to Pic(Sp^G)$$

in general, this is neither injective, nor surjective.

Since Φ^H are symm. monoidal, they take invertible $X \in Sp^G$ to invertible $\Phi^H X \in Sp$, i.e. $\Phi^H X \simeq \mathbb{S}^{n_H}$

This defines a map

$$Pic(Sp^G) \longrightarrow \prod_{\substack{H \in G \\ \text{conj. class}}} \mathbb{Z}$$

$$X \longmapsto n_H$$

tom Dieck–Petrie: • kernel is a finite group
• Image is determined rationally

Ex: D_{2p}

	e	C_2	C_p	D_{2p}
S^1	1	1	1	1
S^σ	1	0	1	0
S^λ	2	1	0	0
?	n	0	0	0

$n = 4$ (and multiples) related to the fact that $H^*(D_{2p})$ is 4-periodic.

III. Isotropy separation

If $X \in Sp^G$ compact, it is dualizable. So we have a map

$$DX \otimes X \to \mathbb{S}$$

X is invertible $\iff$ this map is an equivalence.

We can detect equivalences on geometric fixed points, we get:

Lem: $X \in Sp^G_\omega$, X is invertible $\iff$ $\Phi^H X$ are invertible.

Def. $\mathcal{F} \subseteq Orb(G)$ family if $G/H \in \mathcal{F}$, $H' \subseteq H \Rightarrow G/H' \in \mathcal{F}$.

Functorially have

$$X_{\mathcal{F}} \to X \to X_{\mathcal{F}^c}$$

$X_{\mathcal{F}}$: has vanishing Φ^H for all $G/H \notin \mathcal{F}$

$X_{\mathcal{F}^c}$: has vanishing Φ^H for all $G/H \in \mathcal{F}$.

These are basically never compact (smash with certain $\widetilde{E\mathcal{F}}$)

Def: let $Sp^G_\omega/\mathcal{F}$ be the localization of Sp^G_ω w.r.t. all maps with fiber in $\langle \Sigma^\infty_+ G/H \rangle_{G/H \in \mathcal{F}}$

For $\mathcal{F}' = \mathcal{F} \cup \{G/K\}$, can view

$$\langle \Sigma^\infty_+ G/K \rangle \hookrightarrow Sp^G_\omega/\mathcal{F} \longrightarrow Sp^G_\omega/\mathcal{F}'$$

as a further localization.

In here, $map(\Sigma^\infty_+ G/K, -) = \Phi^K(-)$

so $end(\Sigma^\infty_+ G/K) = \Phi^K \Sigma^\infty_+ G/K = \mathbb{S}[W_G(K)]$

$$\begin{array}{ccccc} \langle \Sigma^\infty_+ G/K \rangle & \to & Sp^G_\omega/\mathcal{F} & \to & Sp^G_\omega/\mathcal{F}' \\ \downarrow & & \downarrow \Phi^K & & \downarrow \\ \langle \mathbb{S}[W_G(K)] \rangle & \to & Mod^{fin}(\mathbb{S}[W_G(K)]) & \to & Mod^{fin}(\mathbb{S}[W_G(K)]) / \langle \mathbb{S}[W_G(K)] \rangle \end{array}$$

$$Mod^{fin}(\mathbb{S}[W_G(K)]) / \langle \mathbb{S}[W_G(K)] \rangle \cong StMod(\mathbb{S}W_G(K))$$

Thm: (K.) This square is a pullback of symm. monoidal ∞-cat.

Thm (K.):

$$\begin{array}{ccc} \mathcal{P}ic(Mod(\mathbb{S}[W_G(K)])) & \to & \mathcal{P}ic(StMod(\mathbb{S}W_G(K))) \\ \downarrow & & \downarrow \\ \mathcal{P}ic(Mod(\mathbb{Z}[W_G(K)])) & \to & \mathcal{P}ic(StMod(\mathbb{Z}W_G(K))) \end{array}$$

is 1-cartesian

Cor:

$$\begin{array}{ccc} \mathcal{P}ic(Sp^G_\omega/\mathcal{F}) & \to & \mathcal{P}ic(Sp^G_\omega/\mathcal{F}') \\ \downarrow & & \downarrow \\ \mathcal{P}ic(Mod(\mathbb{Z}[W_G(K)])) & \to & \mathcal{P}ic(StMod(\mathbb{Z}W_G(K))) \end{array}$$

is 1-cartesian.

Cor. $Pic(Sp^G) = Pic(\mathcal{D}(\mathbb{Z})\text{-valued spectral Mackey functors})$

$Pic(\mathcal{C}) \xrightarrow{\sim} Pic(H\mathbb{Z}\text{-modules in } \mathcal{C})$

R_G [illegible]

$H\mathbb{Z}_G$-modules

$(H\mathbb{Z} \otimes \mathbb{S}_G)$